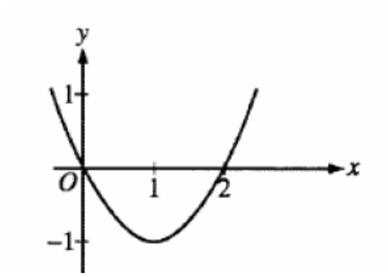
Warm-Up

ELM: Coordinate Geometry & Graphing	Review: Algebra 1 (Standard 16.0)



The figure above shows the graph of $y = f(x)$
What are all values of x for which $f(x) > 0$?

A $x < 0$
B $x > 1$
C $x > 2$
D $0 < x < 2$
E $x < 0$ or $x > 2$

Express the vertex in function notation

Given:
$$f(x) = -x^2 + 3x - 5$$

Find the following function values and write the associated ordered pair:

a) $f(-2)$

b) $f(t+1)$

Current: Algebra 1 (Standard 12.0)	Other: Algebra (Standard 6.0)

Given:

$$g(x) = \frac{x^2 - 9}{x + 3},$$

Write a simpler function that agrees with $g(x)$ at all but one point.

Graph the function that agreed with $g(x)$ in the previous (current) problem.

Standard Definition of Derivative

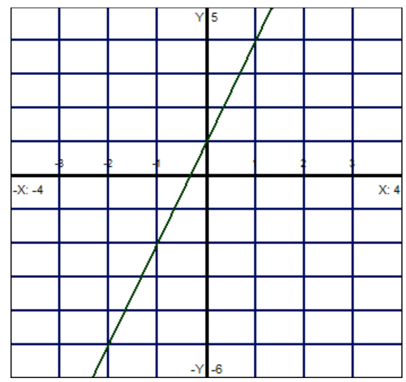
Standards:
Algebra 1 2.0, 4.0, 6.0, 7.0, 11.0, 16.0, 17.0, 18.0
Calculus 4.0

Objectives:
Derive the Standard Definition of Derivative
Connect Algebra standards to Calculus

Consider the function $f(x) = 3x + 1$

Think about what happens to the function values as x gets closer and closer to $x = 1$.

Approach $x = 1$ from the left and approach $x = 1$ from the right.



(Note: x gets closer and closer to $x = 1$ but does not have to reach $x = 1$.)

Notice that the function value, $f(x)$, gets closer and closer to 4 from either side of $x = 1$.

This brings us to the concept if a limit.

For the limit of a function to exist, the left-hand limit must equal the right-hand limit.

The notation is as follows:

The limit of the function f as x approaches c from the right is written $\lim_{x \to c^+} f(x)$

The limit of the function f as x approaches c from the left is written $\lim_{x \to c^-} f(x)$

The limit of the function f as x approaches c is written, $\lim_{x \to c} f(x)$.

$\lim_{x \to c} f(x)$ is said to exist if and only if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$, otherwise the limit Does Not Exist (DNE)

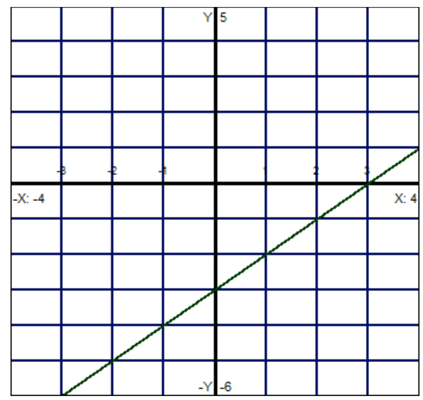
For our example,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$
$$\therefore \lim_{x\to 1} f(x)$$
$$= \lim_{x\to 1}(3x+1)$$
$$= 4$$

Is there a method for finding the limit of this function without using the graph?

Yes, the method is called Direct Substitution.

$$\lim_{x\to 1}(3x+1)$$
$$= 3(1)+1$$
$$= 4$$



Now consider the function $f(x) = x - 3$.

Find $\lim_{x\to -3}(x-3)$.

Using direct substitution,

$$\lim_{x\to -3}(x-3)$$
$$= (-3) - 3$$
$$= -6$$

As x gets closer and closer to $x = -3$, the function value $f(x)$ gets closer and closer to -6.

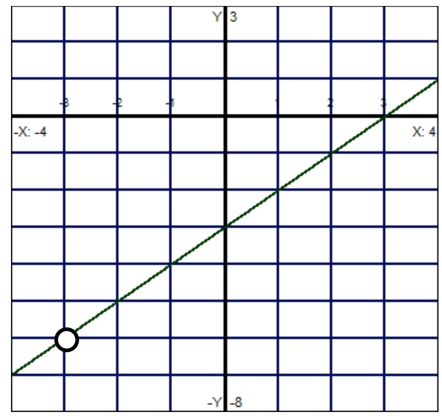
Now let's look at the function $g(x) = \dfrac{x^2-9}{x+3}$. Notice that $x = -3$ is not in the domain of g.

The function can be simplified by factoring the numerator and using the equivalent form of 1.

$$g(x) = \frac{x^2-9}{x+3}$$
$$g(x) = \frac{(x+3)(x-3)}{x+3}$$
$$g(x) = x-3$$

Notice that the simplified function g is the same as our function f above. The original function g agrees with our function f at all but one point. How can we find the point where they do not agree?

Yes! Use limits! Also use the fact that $x = -3$ is not in the domain of *g*. We will use $x = -3$ as our approach value of *x*. A direct substitution in *g* yields $\frac{0}{0}$ indeterminate form so we must simplify *g* then use a direct substitution again.



$$\lim_{x\to -3} \frac{x^2-9}{x+3}$$
$$= \lim_{x\to -3} \frac{(x+3)(x-3)}{x+3}$$
$$= \lim_{x\to -3} (x-3)$$
$$= (-3) - 3$$
$$= -6$$

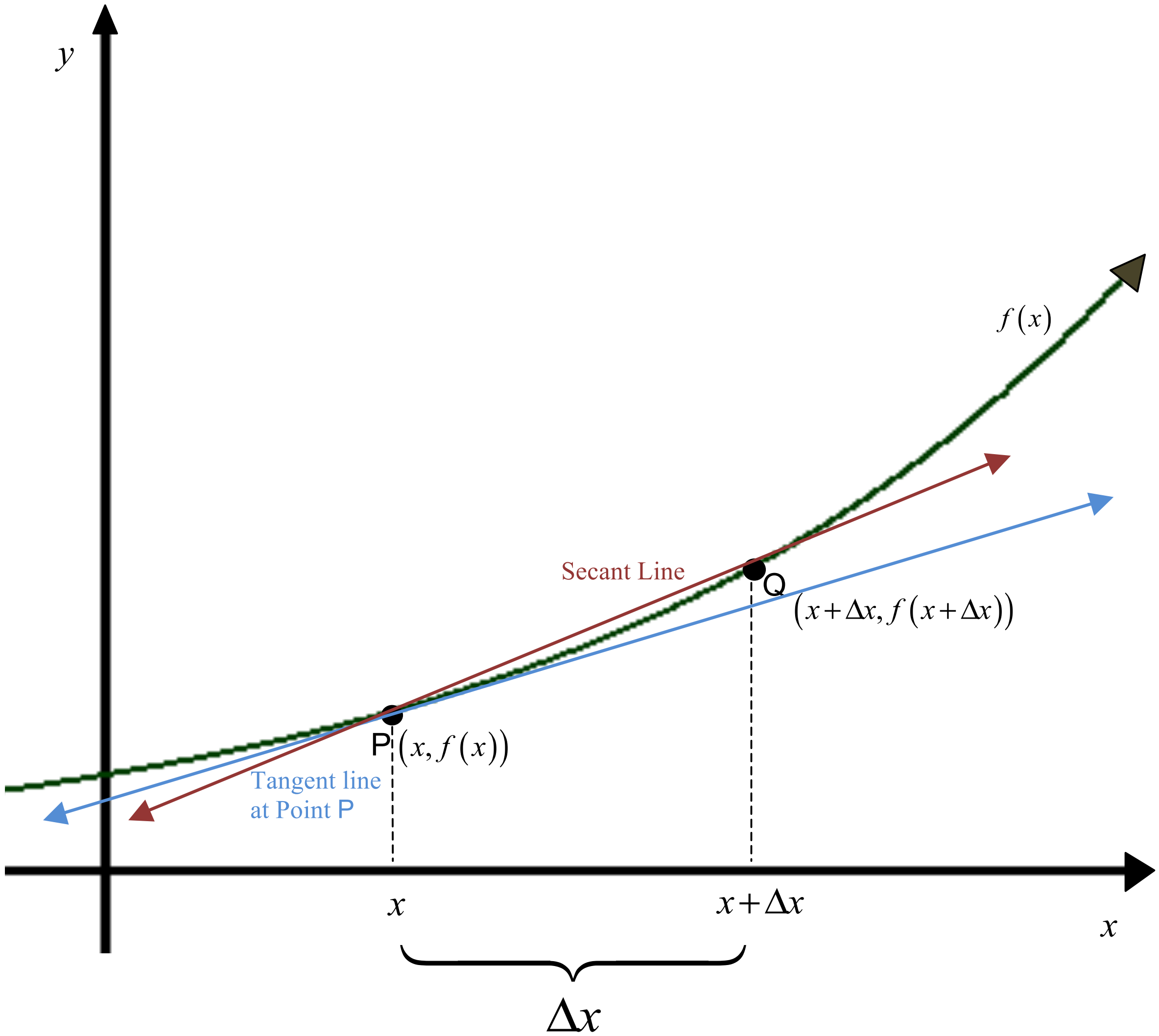
The graphs of *f* and *g* agree at all but one point. The graph of *g* is the line $y = x - 3$ with a hole at the point $(-3,-6)$. Using the limit helps us find the hole in the graph. The equivalent form of 1 lets us know that we have a hole at $x = -3$ and not a vertical asymptote.

Using the following graph discuss:
Average Rate of Change = Slope of Secant Line
Instantaneous Rate of Change = Slope of Tangent Line
This will lead to using the limit and arriving at the

<u>Standard Definition of Derivative</u>

Derivative of a function *f* at *x*

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



In other words, the derivative is the slope of the tangent line to f at a given point.

Notation: $f'(x)$ Read "f prime of x"

Example:

Given $f(x)=x^2$, find $f'(x)$ using the standard definition of derivative.

Method 1:

Using factoring to simplify.

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{2x\Delta x+(\Delta x)^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}(2x+\Delta x)\\
&= 2x+(0)\\
&= 2x\\
&\therefore f'(x)=2x
\end{aligned}
$$

Method 2:

Using fractional decomposition to simplify.

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{2x\Delta x+(\Delta x)^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{2x\Delta x}{\Delta x}+\frac{(\Delta x)^2}{\Delta x}\\
&= \lim_{\Delta x\to 0}(2x+\Delta x)\\
&= 2x+(0)\\
&= 2x\\
&\therefore f'(x)=2x
\end{aligned}
$$

You Try! (you may work with a partner ☺)

1) Given $f(x)=x^2+3x-5$, find $f'(x)$ using the standard definition of derivative.

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^2+3(x+\Delta x)-5-(x^2+3x-5)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2+3x+3\Delta x-5-x^2-3x+5}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{2x\Delta x+(\Delta x)^2+3\Delta x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x+3)}{\Delta x}\\
&= \lim_{\Delta x\to 0}(2x+\Delta x+3)\\
&= 2x+(0)+3\\
&= 2x+3\\
&\therefore f'(x)=2x+3
\end{aligned}
$$

Using factoring to simplify,

2) Given $f(x)=x^3$, find $f'(x)$ using the standard definition of derivative.

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{3x^2\Delta x}{\Delta x}+\frac{3x(\Delta x)^2}{\Delta x}+\frac{(\Delta x)^3}{\Delta x}
\end{aligned}
$$

Using fraction decomposition to simplify.

$$
\begin{aligned}
&= \lim_{\Delta x\to 0}\left(3x^2+3x\Delta x+(\Delta x)^2\right)\\
&= 3x^2+3x(0)+(0)^2\\
&= 3x^2\\
&\therefore f'(x)=3x^2
\end{aligned}
$$

The following question is the type of multiple-choice question that a student would see on the Advanced Placement Exam in Calculus AB.

If $f(x) = 4x^3 + 7x - 5$, which of the following is equivalent to $f'(x)$?

[A] $4x^2 + 7$

[B] $\lim_{\Delta x \to 0} \frac{12(x+\Delta x)^2 + 7 - (12x^2 + 7)}{\Delta x}$

[C] $\lim_{\Delta x \to 0} \frac{4(x+\Delta x)^3 + 7(x+\Delta x) - 5 - (4x^3 + 7x - 5)}{\Delta x}$

[D] $12x^2 + 7x$

The correct answer is [C]